Math 300 April ______ CHANGING VARIABLES IF it has a —

If X is normal, mean 0, s.d. 1, then $W = -X$ is normal, mean 0, s.d. 1

Proof:

Find $F_W(w_0)$: $F_W(w_0) = P(W \le w_0) = P(-X \le w_0) = P(X \ge -w_0) =$

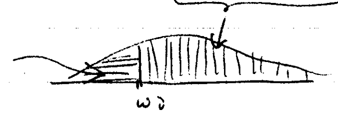
(Way 1)

$= 1 - P(X \le -w_0) = 1 - \int_{-\infty}^{-w_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$ | let $t = -x$, $dt = -dx$, $dx = (-1)dt$, if $x = -\infty$, $t = +\infty$, if $x = -w_0$, $t = +w_0$

$= 1 - \int_{+\infty = t}^{+w_0 = t} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-t)^2}{2}} (-1)\, dt$

$= 1 + \int_{\infty}^{w_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 1 - \underbrace{\int_{w_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}$ (since $\int_a^b = -\int_b^a$)

$= \int_{-\infty}^{w_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$



so $F_W(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, $f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$, and W is Normal, mean 0, s.d. 1

(Way 2, slicker:)

$P(X \ge -w_0) = \int_{-w_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{+w_0 = t}^{-\infty = t} \frac{1}{\sqrt{2\pi}} e^{\frac{-(-t)^2}{2}} (-1)dt$ | let $t = -x$, $dx = (-1)dt$. if $x = \infty$, $t = -\infty$, if $x = -w_0$, $t = w_0$

$= -\int_{w_0}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = +\int_{-\infty}^{w_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, finish as before.

With courage from the simple case, Prove: If X is normal μ, σ and $Y = aX + b$, then Y is normal, mean $(a\mu + \sigma)$, s.d. $|a|\sigma$, in the case $a < 0$.

Proof: Since $a < 0$, $-|a| = a$, switch notations as convenient.

$F_Y(y_0) = P(Y \le y_0) = P(aX+b \le y_0) = P\left(X \ge \frac{y_0 - b}{-|a|}\right) = \int_{\frac{y_0-b}{-|a|}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx =$

Letting $t = ax + b$, $x = \frac{t-b}{-|a|}$, $dx = \frac{1}{-|a|} dt$

If $x = \infty$, $t = -\infty$. If $x = \frac{y_0 - b}{-|a|}$, $t = a\left(\frac{y_0 - b}{a}\right) + b = y_0$

$= \int_{y_0 = t}^{-\infty = t} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{1}{-|a|}\right) dt$ | $\left(\frac{x-\mu}{\sigma}\right)^2 = \left(\frac{\frac{t-b}{-|a|} - \mu}{\sigma}\right)^2 = \left(\frac{t - b - a\mu}{-|a|\sigma}\right)^2 = \left(\frac{t - (a\mu + b)}{|a|\sigma}\right)^2$

$= -\underbrace{\int_{y_0}^{-\infty} \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{|a|\sigma}\right)^2} dt}$

$= +\int_{-\infty}^{y_0} (\quad)\, dt$, so Y is normal, mean $(a\mu + \sigma)$, s.d. $|a|\sigma$

(Way 3) Using Fund. theorem derivative,

$F_Y(y_0) = P\left(X > \frac{y_0 - b}{-|a|}\right) = \int_{\frac{y_0-b}{-|a|}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$

$F_Y'(y) = \frac{d}{dy}\left[-\int^{\frac{y-b}{-|a|}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right] = -\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\frac{y-b}{-|a|}\right)-\mu}{\sigma}\right)^2} \cdot \frac{d}{dy} \cdot \left(\frac{y-b}{-|a|}\right)$

$= \frac{1}{|a|\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{|a|\sigma}\right)^2} = f_Y(y)$