— Algebra of Expectations + Variances —

Read Chapter 7, pp. 220 – 235 (Ash)
Ignore anything with integral signs.

<u>Properties of Expectation p. 220</u> $\underset{(2)}{E(k)=k}$, $\underset{(3)}{E(X+Y)=E(X)+E(Y)}$, $\underset{(4)}{E(kX)=kE(X)}$

We have proved (3) (p. 78 top), and (4) in class.

Proof of (2): $E(k)$ refers to a random variable (call it X) which takes on the single value k, with probability 1. $P(X=k)=1$. $E(X)=k\cdot P(X=k)=k$.

Proof of (5): If <u>X and Y are independent, then $E(XY)=E(X)\cdot E(Y)$</u>.

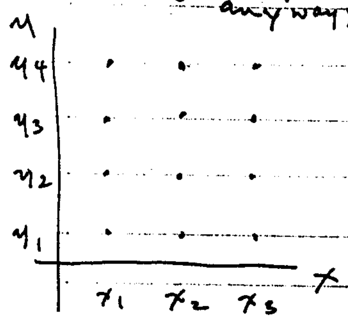
Recall X and Y indep. means $P(X=x \text{ and } Y=y)=P(X=x)\cdot P(Y=y)$.

Now $E(XY)=\sum_{x,y} xy\,P(X=x \text{ and } Y=y)$

independence

$$=\sum_{x,y} xy\,P(X=x)\cdot P(Y=y)$$

Summing over all points any way you like



$$=\sum_{x,y}\left[\left(x\,P(X=x)\right)\cdot\left(y\,P(Y=y)\right)\right]$$

$$=\sum_{x=x_1}^{x_m}\sum_{y=y_1}^{y_m}\left[x_iP(X=x_i)\cdot y_jP(Y=y_j)\right]$$

Then add those for all x's. sum down each column.

$$=\sum_{x=x_1}^{x_m}\left[\underline{x_iP(X=x_i)}\sum_{y=y_1}^{y_m}y_jP(Y=y_j)\right]$$

this is constant within a single column.

$$=\sum_{x=x_1}^{x_m}\left[x_i\,P(X=x_i)\cdot E(Y)\right]$$

a number – constant

$$\cancel{=E(Y)\cdot\sum_{x} x_iP(X=x_i)}$$

$$=E(Y)\cdot E(X)=E(X)\cdot E(Y)$$

Let $x_iP(X=x_i)=a_i$
$y_jP(Y=y_j)=b_j$
$\sum_{i,j}[a_ib_j]$
$=\sum_i\sum_j a_ib_j$
$=\sum_i a_i\sum_j b_j$
(Let $\sum_j b_j=B$)
$=\sum_i a_iB$
$=B\sum_i a_i$
(Let $\sum a_i=A$)
$=BA$.

<u>Var</u> $X=E(X^2)-(EX)^2$ p225. <u>Know proof!</u>

$E(X^2)-\mu_X^2$ $\left(=E(X-\mu_X)^2\right)$ - a remarkable "coincidence"

Note $E(X^2)=\mathrm{Var}\,X+\mu_X^2$

[$\mathrm{Cov}(X,Y)=E(X-\mu_X)(Y-\mu_Y)=E(XY)-\mu_X\mu_Y$ p.226 - <u>know proof</u>.
If X and Y are <u>indep</u>, $E(XY)=E(X)\cdot E(Y)=\mu_X\mu_Y$, so $\mathrm{Cov}(X,Y)=0$.

$\mathrm{Var}(X+Y)=\mathrm{Var}\,X+\mathrm{Var}\,Y+2\,\mathrm{cov}(X,Y)$ (p.227 - version for n) - see proof for 2, next p.

<u>If X and Y are indep. then $\mathrm{Var}(X+Y)=\mathrm{var}\,X+\mathrm{var}\,Y$</u> (since $\mathrm{cov}(X,Y)=0$)

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$

Proof: Ash does it using $E(X^2) - \mu_X^2$ for $\mathrm{Var}(X)$. I'll use the def. of Var.

Let $W = X+Y$. $\mu_W = E(W) = E(X+Y) = \mu_X + \mu_Y$.

$\mathrm{Var}(X+Y) = \mathrm{Var}(W) = E(W-\mu_W)^2$

$= E[(X+Y) - (\mu_X+\mu_Y)]^2 = E[(X-\mu_X) + (Y-\mu_Y)]^2$ $\qquad (a+b)^2 = a^2 + 2ab + b^2$

$= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2]$

$= E(X-\mu_X)^2 + 2E[(X-\mu_X)(Y-\mu_Y)] + E(Y-\mu_Y)^2$

$= \mathrm{var}\,X + 2\,\mathrm{Cov}(X,Y) + \mathrm{var}\,Y$

For more than 2, you have to keep track of the cross-products.

p228
- $\mathrm{Var}\,k = 0$, and if $\mathrm{Var}\,X = 0$, X is constant
- $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}\,X$
- $\mathrm{Var}(-X) = \mathrm{Var}\,X$, so if X & Y are indep, $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Mean + Variance of Binomial pp 229-30. Moore does the same.
$np \quad npq$

ex. 2 p230 Variance of hypergeometric (I'll do derivation)

Draw n from N; D "defectives". $X = \#$ of successes (defectives)

$E(X) = n \cdot \frac{D}{N}$, drawing with or without replacement.

$\mathrm{Var}\,X = n \cdot \frac{D}{N} \cdot \left(\frac{N-D}{N}\right) \cdot \left(\frac{N-n}{N-1}\right)$ without replacement.

If we let $p = \frac{D}{N}$, $q = \frac{N-D}{N}$,

$\mathrm{Var}\,X = npq \cdot \left(\frac{N-n}{N-1}\right)$, "finite population correction".

p235 $\frac{\mathrm{Cov}(X,Y)}{\sigma_X\,\sigma_Y} = \rho(X,Y)$ – "correlation of X,Y" – theoretical version of r.

HW: 7-1 (p.223) #14. 7-2 (p.233) 1, 3, 6a, 7, 8, 9 (proofs!) 10 (use $E(X^2) = \mathrm{Var}\,X + \mu_X^2$)

To be assigned: 13, 14 (check Var X with Hypergeom. formula), 16, (16/17 optional) 19b (use p.82 + $\mathrm{Var(geom)} = \frac{q}{p^2}$)

a) Show how to find $\mathrm{Var}(X)$ if you know $E(X(X-1))$ and $E(X)$. (Use Ch 7-1 (3) + 7-2 (3))

b) For the Poisson distribution, find $E(X(X-1))$, then Var X

x	0	1	2	3
$P(X=x)$	$e^{-\lambda}$	$\frac{e^{-\lambda}\lambda}{1}$	?	?
$x(x-1)$	0	0	$2\cdot1$	$3\cdot2$
$x(x-1)\cdot P(X=x)$	0	0	?	?